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Exponential Convergence for the k-th Order Statistics

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Abstract. Let $X_1, X_2, \cdots, X_n$ be the samples of an arbitrary population having a uniform distribution on the interval $[0, 1]$ and $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ denote the order statistics. The moderate deviations, large deviations and Cramér large deviations for the k-th order statistics $X_{k,n}$ are established. Furthermore, we also discuss the case that the samples come from the general distribution F.

1. The First Section

Suppose that we have an independent and identically distributed sample of size n from a distribution function F. Let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ denote the order statistics of $X_1, \cdots, X_n$ and let ξ_p denote an p-th quantile of F, i.e.,

$$\xi_p = \inf\{x : F(x) \ge p\}, \quad p \in (0, 1).$$

It is well-known that, under the above definition, ξ_p is unique. If $p = 1/2$, then ξ_p is the median of F.

Order statistics of time series data play an important role in robust statistical inference about various process parameters, particularly when the underlying distribution is heavy-tailed. It is well known that financial time series data often are heavy-tailed. As a result, quantile based methods are being increasingly developed and employed in diverse problems in finance, such as, quantile-hedging, optimal portfolio allocation, risk management, and so on.

There are numerous literatures to study the order statistics. Asymptotic normality and consistency of various forms about order statistics under some regular conditions are well known. Adler [1, 2] obtained some limit theorems for maximal and minimal order statistics. Park [11, 12] studied the asymptotic Fisher information in order statistics. Suppose $0 < p < 1$, $p \le (k/n) \le p + (1/n)$ for $n \to \infty$, $k \to \infty$, then in the latter limit $X_{k,n}$ converges in probability to ξ_p (see [4, 15]). In addition, if F has a continuous first derivative f in the neighborhood of ξ_p and $f(\xi_p) > 0$, then

$$\frac{\sqrt{n} f(\xi_p)(X_{k,n} - \xi_p)}{\sqrt{p(1-p)}} \xrightarrow{d} N(0, 1), \quad \text{as } n \to \infty,$$

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where $N(0,1)$ denotes the standard normal random variable (see [4, 10, 15]). Assume that $F(x)$ is twice differentiable at ξ_p, with $F^{'}(\xi_p) = f(\xi_p) > 0$. If $k = np + o(\sqrt{n}(\log n)^{\delta})$ for some $\delta \geq 2$, then Bahadur [3] proved

$$X_{k,n} = \xi_p + \frac{k/n - \sum_{i=1}^{n} I_{\{X_i \leq \xi_p\}}}{f(\xi_p)} + R_n, \text{ a.e.}$$

where $R_n = O\left(n^{-3/4}(\log n)^{(1/2)(\delta+1)}\right)$, a.e. as $n \to \infty$. In his paper, he also raised the question of finding the exact order of R_n. Further analysis by Eicker [7] revealed that $R_n = o_p(n^{-3/4}g(n))$ if and only if $g(n) \to \infty$. Kiefer [8] obtained very precise details. Lahiri and Sun [9] gave a Berry-Esseen theorem for sample quantiles of strongly-mixing random variables under a polynomial mixing rate. Very recently, Xu and Miao [17] proved the large and moderate deviation principle and Bahadur's asymptotic efficiency of the deviation between sample quantiles and p-quantile under some weak conditions.

In [6], Egorov and Nevzorov gave the rate of convergence to the normal law of the properly centered and normalized sums $\sum_{i=1}^{n} c_i X_{i,n}$ with arbitrary coefficients $c_i (1 \leq i \leq n)$. They also obtained as a particular case of a more general result, the following estimate for uniform order statistics:

$$\sup_x \left| P\left\{ X_{i,n} - \frac{i}{n+1} \leq x\beta_2 \right\} - \Phi(x) \right| \leq C\left(\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n-i+1}} \right),$$

where $\beta_2 = i(n-i+1)/(n+1)^2(n+2)$ is the variance of the beta distribution with parameters i and $n-i+1$ and C is a constant. In [16], Wellner obtained a law of the iterated logarithm for linear combinations of order statistic.

Motivated by these works, in present paper, we discuss the exponential convergence (i.e., moderate and large deviations) for the k-th order statistics. Firstly, we discuss the simple case that the samples come from a uniform distribution on $[0,1]$. Let $X_1, X_2, \cdots, X_n$ be the samples of an arbitrary population having a uniform distribution on the interval $[0,1]$ and $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$ denote the order statistics. Although the samples from $[0,1]$ seems very simple, it is very important in practical application. For instance, it can be used to simulate a random variable as a general method. From the inverse transformation method (cf. [14]), it is well known that, for any continuous distribution function F, the random variable $X = F^{-1}(U)$ has distribution function F, where U is the uniform $[0,1]$ random variable. Hence we can simulate a random variable X from the continuous distribution F, when F^{-1} is computable, by simulating a random number U and then setting $X = F^{-1}(U)$. By using the methods to study the uniform samples, we can also obtain some results for the case that the samples come from a general distribution. The paper is organized as follows. The main results are stated in Section 2 and we give their proofs in the last section.

2. Main Results

2.1. Moderate Deviations

In the subsection we will show the following moderate deviations for the k-th order statistics.

Theorem 2.1. *Let $1 \leq k = k(n) \leq n$ and let σ_n, b_n be two positive real sequences satisfying*

$$\lim_{n\to\infty} \frac{b_n^2}{\sigma_n^2} = 0, \quad \lim_{n\to\infty} \frac{b_n \sigma_n}{n} = 0, \tag{1}$$

and

$$\lim_{n\to\infty} \left(\frac{k - k^2/n}{\sigma_n^2} \right) = \sigma^2 \in (0,1). \tag{2}$$

Then for any $r > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log P\left(\frac{n}{b_n \sigma_n} \left| X_{k,n} - \frac{k}{n} \right| \geq r \right) = -\frac{r^2}{2\sigma^2}.$$

The condition (1) implies

$$\lim_{n\to\infty} \frac{b_n^2}{n} = 0.$$

If we take $\sigma_n^2 = n$ in Theorem 2.1, then we have the following

Corollary 2.2. *Let $1 \le k = k(n) \le n$ and let b_n be a positive real sequences satisfying*

$$\lim_{n\to\infty} \frac{b_n^2}{n} = 0, \tag{3}$$

and

$$\lim_{n\to\infty} \left(\frac{k}{n} - \frac{k^2}{n^2}\right) = \sigma^2 \in (0,1). \tag{4}$$

Then for any $r > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log P\left(\frac{\sqrt{n}}{b_n}\left|X_{k,n} - \frac{k}{n}\right| \ge r\right) = -\frac{r^2}{2\sigma^2}.$$

For the sample medians, one can obtain the following

Corollary 2.3. *Let b_n be a positive real sequences satisfying*

$$\lim_{n\to\infty} \frac{b_n^2}{n} = 0. \tag{5}$$

Then for any $r > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log P\left(\frac{\sqrt{n}}{b_n}\left|X_{\lfloor \frac{n}{2} \rfloor,n} - \frac{1}{2}\right| \ge r\right) = -2r^2,$$

where $\lfloor \frac{n}{2} \rfloor$ denotes the integral part of $\frac{n}{2}$.

2.2. *Large Deviations*

In the subsection we discuss the large deviations and Cramér large deviations for the k-th order statistics.

Theorem 2.4. *Let $1 \le k = k(n) \le n$ with*

$$\lim_{n\to\infty} \frac{k}{n} = \gamma \in (0,1). \tag{6}$$

Then we have for any $\gamma < r < 1$,

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(X_{k,n} > r\right) = (1-\gamma)\log\frac{1-r}{1-\gamma} + \gamma \log \frac{r}{\gamma}$$

and for any $0 < r < \gamma$,

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(X_{k,n} < r\right) = (1-\gamma)\log\frac{1-r}{1-\gamma} + \gamma \log \frac{r}{\gamma}.$$

Corollary 2.5. *Let $1 \le k = k(n) \le n$ with*

$$\lim_{n\to\infty} \frac{k}{n} = \frac{1}{2}. \tag{7}$$

Then for any $1/2 < r < 1$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(X_{\lfloor \frac{n}{2} \rfloor,n} > r\right) = \log 2\sqrt{r(1-r)}$$

and for any $0 < r < 1/2$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(X_{\lfloor \frac{n}{2} \rfloor,n} < r\right) = \log 2\sqrt{r(1-r)}.$$

The following result is the Cramér large deviation of the order statistics.

Theorem 2.6. *Let* $r \in (0,1)$, $1 \le k = k(n) \le n$, *and*

$$x_n = \frac{r - k/n}{\sqrt{r(1-r)}}\sqrt{n}.$$

If $k/n \uparrow r$, *then we have*

$$\frac{P(X_{k,n} > r)}{P(N > x_n)} = \exp\left\{\frac{x_n^3}{\sqrt{n}}\lambda_1\left(\frac{x_n}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{x_n+1}{\sqrt{n}}\right)\right] \tag{8}$$

and if $k/n \downarrow r$, *then we have*

$$\frac{P(X_{k,n} < r)}{P(N > x_n)} = \exp\left\{\frac{x_n^3}{\sqrt{n}}\lambda_2\left(\frac{x_n}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{x_n+1}{\sqrt{n}}\right)\right]. \tag{9}$$

Here N denotes the standard normal random variable and

$$\lambda_1(t) = \sum_{k=0}^{\infty} c_{1,k}t^k \quad \lambda_2(t) = \sum_{k=0}^{\infty} c_{2,k}t^k$$

are power series with coefficients depending only on the cumulants of the random variable

$$\eta_1 := \mathbb{I}_{\{X_1 > r\}} - (1-r), \quad \eta_2 := \mathbb{I}_{\{X_1 < r\}} - r$$

which converges for sufficiently small values of $|t|$.

From Theorem 2.6, we have the following results.

Theorem 2.7. *Let* $r \in (0,1)$, $1 \le k = k(n) \le n$. *If* $k/n \uparrow r$, *then we have*

$$\lim_{n\to\infty} \frac{1}{n}\log P(X_{k,n} > r) = 0 \tag{10}$$

and if $k/n \downarrow r$, *then we have*

$$\lim_{n\to\infty} \frac{1}{n}\log P(X_{k,n} < r) = 0. \tag{11}$$

2.3. Further Discussions

In this subsection, we consider the general case that $X_1, X_2, \cdots, X_n$ are the samples of an arbitrary population having distribution F on $\mathbb{R}$ and $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ denote the order statistics. Then we have the following results.

Theorem 2.8. *Let* $1 \le k = k(n) \le n$ *with*

$$\lim_{n\to\infty} \frac{k}{n} = \gamma \in (0,1). \tag{12}$$

Then we have for $0 < \gamma < F(r) < 1$,

$$\lim_{n\to\infty} \frac{1}{n}\log P(X_{k,n} > r) = (1-\gamma)\log\frac{1-F(r)}{1-\gamma} + \gamma\log\frac{F(r)}{\gamma}$$

and for $0 < F(r) < \gamma < 1$,

$$\lim_{n\to\infty} \frac{1}{n}\log P(X_{k,n} < r) = (1-\gamma)\log\frac{1-F(r)}{1-\gamma} + \gamma\log\frac{F(r)}{\gamma}.$$

Corollary 2.9. *Let* $1 \le k = k(n) \le n$ *with*

$$\lim_{n\to\infty} \frac{k}{n} = \frac{1}{2}. \tag{13}$$

Then for any $1/2 < F(r) < 1$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(X_{\lfloor \frac{n}{2} \rfloor, n} > r\right) = \log 2\sqrt{F(r)(1-F(r))}$$

and for any $0 < F(r) < 1/2$ *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(X_{\lfloor \frac{n}{2} \rfloor, n} < r\right) = \log 2\sqrt{F(r)(1-F(r))}.$$

Theorem 2.10. *Let* $r \in (0,1)$, $1 \le k = k(n) \le n$, *and*

$$x_n^{'} = \frac{F(r) - k/n}{\sqrt{F(r)(1-F(r))}}\sqrt{n}.$$

If $k/n \uparrow F(r)$, *then we have*

$$\frac{P(X_{k,n} > r)}{P(N > x_n^{'})} = \exp\left\{\frac{x_n^{'3}}{\sqrt{n}}\lambda_1^{'}\left(\frac{x_n^{'}}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{x_n^{'}+1}{\sqrt{n}}\right)\right] \tag{14}$$

and if $k/n \downarrow F(r)$, *then we have*

$$\frac{P(X_{k,n} < r)}{P(N > x_n^{'})} = \exp\left\{\frac{x_n^{'3}}{\sqrt{n}}\lambda_2^{'}\left(\frac{x_n^{'}}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{x_n^{'}+1}{\sqrt{n}}\right)\right]. \tag{15}$$

Here N *denotes the standard normal random variable and*

$$\lambda_1^{'}(t) = \sum_{k=0}^{\infty} c_{1,k} t^k \quad \lambda_2^{'}(t) = \sum_{k=0}^{\infty} c_{2,k} t^k$$

are power series with coefficients depending only on the cumulants of the random variable

$$\eta_1 := \mathbb{I}_{\{X_1 > r\}} - (1 - F(r)), \quad \eta_2 := \mathbb{I}_{\{X_1 < r\}} - F(r)$$

which converges for sufficiently small values of $|t|$.

From Theorem 2.10, we have the following results.

Theorem 2.11. *Let* $r \in (0,1)$, $1 \le k = k(n) \le n$. *If* $k/n \uparrow F(r)$, *then we have*

$$\lim_{n\to\infty} \frac{1}{n} \log P(X_{k,n} > r) = 0 \tag{16}$$

and if $k/n \downarrow F(r)$, *then we have*

$$\lim_{n\to\infty} \frac{1}{n} \log P(X_{k,n} < r) = 0. \tag{17}$$

3. Proofs of Main Results

Proof of Theorem 2.1. For any $r > 0$, we need only give the proof of the following form

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log P\left(\frac{n}{b_n\sigma_n}\left(X_{k,n} - \frac{k}{n}\right) \geq r\right) = -\frac{r^2}{2\sigma^2}, \tag{18}$$

and the other form is similar. Since

$$\begin{aligned}
&P\left(\frac{n}{b_n\sigma_n}\left(X_{k,n} - \frac{k}{n}\right) \geq r\right)\\
=&P\left(\sum_{k=1}^{n} \mathbb{I}_{\{X_k > \frac{rb_n\sigma_n}{n} + \frac{k}{n}\}} > n-k\right)\\
=&P\left(\frac{1}{b_n\sigma_n}\sum_{k=1}^{n}\left[\mathbb{I}_{\{X_k > \frac{rb_n\sigma_n}{n} + \frac{k}{n}\}} - \left(1 - \frac{rb_n\sigma_n}{n} - \frac{k}{n}\right)\right] > r\right)\\
=:&P\left(\frac{1}{b_n\sigma_n}\sum_{k=1}^{n} Z_{k,n} > r\right),
\end{aligned}$$

then the Cramér functional of $\frac{1}{b_n\sigma_n}\sum_{k=1}^{n} Z_{k,n}$ is

$$\begin{aligned}
\Lambda(\lambda) :=& \lim_{n\to\infty} \frac{1}{b_n^2} \log E\exp\left(\lambda\frac{b_n}{\sigma_n}\sum_{k=1}^{n} Z_{k,n}\right)\\
=& \lim_{n\to\infty} \frac{1}{b_n^2}\sum_{k=1}^{n} \log\left(1 + \frac{\lambda^2 b_n^2}{2\sigma_n^2} EZ_{k,n}^2 + o\left(\frac{b_n^2}{\sigma_n^2}\right)\right) \quad \text{for } \lambda \in \mathbb{R}.
\end{aligned}$$

Because of

$$EZ_{k,n}^2 = \left(\frac{rb_n\sigma_n}{n} + \frac{k}{n}\right)\left(1 - \frac{rb_n\sigma_n}{n} - \frac{k}{n}\right),$$

we have for $\lambda \in \mathbb{R}$,

$$\Lambda(\lambda) = \lim_{n\to\infty} \frac{1}{b_n^2} \log\left(1 + \frac{\lambda^2 b_n^2}{2n\sigma_n^2}\left(k - k^2/n\right) + o\left(\frac{b_n^2}{\sigma_n^2}\right)\right)^n = \frac{\lambda^2\sigma^2}{2}.$$

By the Gärtner-Ellis theorem (cf. [5]), (18) can be obtained. Thus the proof of the theorem can be completed. □

Proof of Theorem 2.4. For any $\gamma < r < 1$, we have

$$P(X_{k,n} > r) = P\left(\frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_{\{X_k > r\}} > 1 - \frac{k}{n}\right).$$

Now we calculate the Cramér functional of $\frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_{\{X_k > r\}}$, i.e., for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\Lambda(\lambda) :=& \lim_{n\to\infty} \frac{1}{n} \log E\exp\left(\lambda\sum_{k=1}^{n} \mathbb{I}_{\{X_k > r\}}\right)\\
=& \log E\exp\left(\lambda\mathbb{I}_{\{X_1 > r\}}\right) = \log\left(r + e^{\lambda}(1-r)\right), \quad \text{for } \lambda \in \mathbb{R}.
\end{aligned}$$

which can yields

$$\begin{aligned}\Lambda^*(x) :=& \sup\{\lambda x - \Lambda(\lambda)\}\\ =& x\log\left(\frac{rx}{(1-x)(1-r)}\right) - \log\left(\frac{r}{1-x}\right)\\ =& x\log\frac{x}{1-r} - (1-x)\log\frac{r}{1-x}, \quad x\in[0,1],\end{aligned}$$

and $\Lambda^*(x) = \infty$ otherwise. Hence we have for $0 < \gamma < r < 1$,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log P(X_{k,n} > r) &= \lim_{n\to\infty}\frac{1}{n}\log P\left(\frac{1}{n}\sum_{k=1}^{n}\mathbb{I}_{\{X_k>r\}} > 1-\frac{k}{n}\right)\\ &= -\inf_{x\geq 1-\gamma}\Lambda^*(x) = (1-\gamma)\log\frac{1-r}{1-\gamma} + \gamma\log\frac{r}{\gamma}.\end{aligned}$$

For the case $0 < r < \gamma$, by the same proof, the desired result can be obtained. □

Proof of Theorem 2.6. It is easy to see

$$P(X_{k,n} > r) = P\left(\frac{1}{\sqrt{nr(1-r)}}\sum_{k=1}^{n}(\mathbb{I}_{\{X_k>r\}} - (1-r)) > \frac{r-k/n}{\sqrt{r(1-r)}}\sqrt{n}\right).$$

Let

$$Z_n = \frac{1}{\sqrt{nr(1-r)}}\sum_{k=1}^{n}(\mathbb{I}_{\{X_k>r\}} - (1-r)),\ x_n = \frac{r-k/n}{\sqrt{r(1-r)}}\sqrt{n},$$

then from Theorem 5.23 in [13], we get

$$\frac{P(Z_n > x_n)}{P(N > x_n)} = \exp\left\{\frac{x_n^3}{\sqrt{n}}\lambda_1\left(\frac{x_n}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{x_n+1}{\sqrt{n}}\right)\right].$$

If we note that

$$P(X_{k,n} < r) = P\left(\frac{1}{\sqrt{nr(1-r)}}\sum_{k=1}^{n}(\mathbb{I}_{\{X_k<r\}} - r) > \frac{k/n - r}{\sqrt{r(1-r)}}\sqrt{n}\right),$$

then from Theorem 5.23 in [13], the equality (9) holds. □

Proof of Theorem 2.8. For the general case, we have the following Cramér functional

$$\begin{aligned}\Lambda(\lambda) :=& \lim_{n\to\infty}\frac{1}{n}\log E\exp\left(\lambda\sum_{k=1}^{n}\mathbb{I}_{\{X_k>r\}}\right)\\ =& \log E\exp(\lambda\mathbb{I}_{\{X_1>r\}}) = \log\left(F(r) + e^{\lambda}(1-F(r))\right),\end{aligned}$$

which can yields

$$\begin{aligned}\Lambda^*(x) :=& \sup\{\lambda x - \Lambda(\lambda)\}\\ =& x\log\frac{xF(r)}{(1-x)(1-F(r))} - \log\frac{F(r)}{1-x}\\ =& x\log\frac{x}{1-F(r)} - (1-x)\log\frac{F(r)}{1-x}, \quad x\in[0,1],\end{aligned}$$

and $\Lambda^*(x) = \infty$ otherwise. Hence we have for $0 < \gamma < F(r) < 1$,

$$\lim_{n\to\infty} \frac{1}{n} \log P(X_{k,n} > r) = -\inf_{x\geq 1-\gamma} \Lambda^*(x)$$
$$=(1-\gamma)\log\frac{1-F(r)}{1-\gamma} + \gamma\log\frac{F(r)}{\gamma}.$$

For the case $0 < F(r) < \gamma$, by the same proof, the desired result can be obtained. □
Proof of Theorem 2.10. It is easy to see

$$P(X_{k,n} > r) = P\left(\frac{\sum_{k=1}^{n}(\mathbb{I}_{\{X_k>r\}} - (1-F(r)))}{\sqrt{nF(r)(1-F(r))}} > \frac{F(r)-k/n}{\sqrt{F(r)(1-F(r))}}\sqrt{n}\right).$$

Let

$$Z_n' = \frac{\sum_{k=1}^{n}(\mathbb{I}_{\{X_k>r\}} - (1-F(r)))}{\sqrt{nF(r)(1-F(r))}},\ x_n' = \frac{F(r)-k/n}{\sqrt{F(r)(1-F(r))}}\sqrt{n},$$

then from Theorem 5.23 in [13], we get

$$\frac{P(Z_n' > x_n')}{P(N > x_n')} = \exp\left\{\frac{x_n'^3}{\sqrt{n}}\lambda_1\left(\frac{x_n'}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{x_n'+1}{\sqrt{n}}\right)\right].$$

If we note that

$$P(X_{k,n} < r) = P\left(\frac{\sum_{k=1}^{n}(\mathbb{I}_{\{X_k<r\}} - F(r))}{\sqrt{nF(r)(1-F(r))}} > \frac{k/n-F(r)}{\sqrt{F(r)(1-F(r))}}\sqrt{n}\right),$$

then from Theorem 5.23 in [13], the equality (15) holds. □

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